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Pointwise Convergence of the Fejér Means of Functions on Unbounded Vilenkin Groups*

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In this paper we prove that for a function $f \in L^p(G_m)$ where $1 < p$ the Fejér means $\sigma_n f$ converge to f almost everywhere with respect to the character system of any (bounded or not) Vilenkin group G_m. © 1999 Academic Press

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1. INTRODUCTION AND THE MAIN THEOREM

One of the most celebrated problems in dyadic harmonic analysis is the pointwise convergence of the Fejér (or $(C, 1)$) means of functions on unbounded Vilenkin groups. We give a partial answer to this question. Namely, we prove that if $f \in L^p(G_m)$ for $p > 1$, then $\sigma_n f \to f$ almost everywhere. First we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced by Vilenkin in 1947 (see, e.g., [Vil, AVD]). Let $m := (m_k, k \in \mathbf{N})$ ($\mathbf{N} := \{0, 1, ...\}$ be a sequence of integers, each of which is not less than 2. Let Z_{m_k} denote the m_kth discrete cyclic group. Z_{m_k} can be represented by the set $\{0, 1, ..., m_k - 1\}$, where the group operation is the mod m_k addition and every subset is open. The measure on Z_{m_k}, μ_k is defined such that the measure of every singleton is $1/m_k$ ($k \in \mathbf{N}$). Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

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Then every $x \in G_m$ can be represented by a sequence $x=(x_i, i \in \mathbf{N})$, where $x_i \in Z_{m_i}$ $(i \in \mathbf{N})$. The group operation on G_m (denoted by $+$) is the coordinatewise addition (the inverse operation is denoted by $-$), the measure (denoted by μ) and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n \in \mathbf{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group.

A base for the neighborhoods of G_m can be given as

$$I_0(x) := G_m, \qquad I_n(x) := \{ y=(y_i, i \in \mathbf{N}) \in G_m : y_i = x_i \text{ for } i<n \}$$

for $x \in G_m$, $n \in \mathbf{P} := \mathbf{N} \backslash \{0\}$. Let $0=(0, i \in \mathbf{N}) \in G_m$ denote the null element of G_m, $I_n := I_n(0)$ $(n \in \mathbf{N})$. Furthermore, let $L^p(G_m)$ $(1 \leqslant p \leqslant \infty)$ denote the usual Lebesgue spaces ($\|\cdot\|_p$ the corresponding norms) on G_m, $\mathscr{A}_n$, the σ algebra generated by the sets $I_n(x)$ $(x \in G_m)$, and E_n the conditional expectation operator with respect to $\mathscr{A}_n$ $(n \in \mathbf{N})$ $(E_{-1} f := 0$ $(f \in L^1)$.)

The concept of the maximal Hardy space ([SWS]) $H^1(G_m)$ is defined by the maximal function $f^* := \sup_n |E_n f|$ $(f \in L^1(G_m))$, saying that f belongs to the Hardy space $H^1(G_m)$ if $f^* \in L^1(G_m)$. $H^1(G_m)$ is a Banach space with the norm

$$\|f\|_{H^1} := \|f^*\|_1.$$

Let $M_0 := 1$, $M_{n+1} := m_n M_n$ $(n \in \mathbf{N})$. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \qquad (n_i \in \{0, 1, ..., m_i - 1\}, i \in \mathbf{N}),$$

where only a finite number of n_i's differ from zero. The generalized Rademacher functions are defined as

$$r_n(x) := \exp\left(2\pi\imath \frac{x_n}{m_n}\right) \qquad (x \in G_m, n \in \mathbf{N}, \imath := \sqrt{-1}).$$

It is known that

$$\sum_{i=0}^{m_n - 1} r_n^i(x) = \begin{cases} 0, & \text{if } \ x_n \neq 0 \\ m_n, & \text{if } \ x_n = 0 \end{cases}, \quad (x \in G_m, n \in \mathbf{N}).$$

The nth Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \qquad (n \in \mathbf{N}).$$

The system $\psi := (\psi_n : n \in \mathbf{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m and all the characters of G_m are of this form. Define the m-adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\bmod m_j))\, M_j \qquad (k, n \in \mathbf{N}).$$

Then, $\psi_{k \oplus n} = \psi_k \psi_n$, $\psi_n(x+y) = \psi_n(x)\, \psi_n(y)$, $\psi_n(-x) = \bar{\psi}_n(x)$, $|\psi_n| = 1$ $(k, n \in \mathbf{N},\ x, y \in G_m)$.

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the Vilenkin system ψ as

$$\hat{f}(n) := \int_{G_m} f \bar{\psi}_n \qquad S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\, \psi_k,$$

$$D_n(y, x) = D_n(y - x) := \sum_{k=0}^{n-1} \psi_k(y)\, \bar{\psi}_k(x),$$

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f, \qquad K_n(y, x) = K_n(y - x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(y - x),$$

$$\left(n \in \mathbf{P},\ y, x \in G_m,\ \hat{f}(0) := \int_{G_m} f,\ S_0 f = D_0 = 0,\ f \in L^1(G_m) \right).$$

It is well known that

$$(S_n f)(y) = \int_{G_m} f(x)\, D_n(y - x)\, dx,$$

$$(\sigma_n f)(y) = \int_{G_m} f(x)\, K_n(y - x)\, dx \qquad (n \in \mathbf{P},\ y \in G_m,\ f \in L^1(G_m)).$$

It is also well known that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n(0) \\ 0 & \text{if } x \notin I_n(0) \end{cases},$$

$$S_{M_n} f(x) = M_n \int_{I_n(x)} f = E_n f(x) \qquad (f \in L^1(G_m),\ n \in \mathbf{N}).$$

Moreover [AVD], for $n \in \mathbf{P}$,

$$D_n = \psi_n \sum_{j=0}^{\infty} D_{M_j} \sum_{i=m_j - n_j}^{m_j - 1} r_j^i. \tag{1.1}$$

That is, for $z \in I_t \backslash I_{t+1}$ $(t \in \mathbf{N})$,

$$D_n(z) = \psi_n(z) \left(\sum_{j=0}^{t-1} n_j M_j + M_t \sum_{i=m_t - n_t}^{m_t - 1} r_t^i(z) \right). \tag{1.2}$$

Define the maximal operator

$$\sigma^* f := \sup_{n \in \mathbf{P}} |\sigma_n f|.$$

Let X and Y be either $H^1(G_m)$ or $L^p(G_m)$ for some $1 \leqslant p \leqslant \infty$ with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. We say that operator σ is of type (Y, X) if there exist an absolute constant $c > 0$ for which $\|\sigma^* f\|_Y \leqslant c \|f\|_X$ for all $f \in X$. σ^* is of weak type (L^1, L^1) if there exist an absolute constant $c > 0$ for which $\mu(\sigma^* f > \lambda) \leqslant c \|f\|_1 / \lambda$ for all $\lambda > 0$ and $f \in L^1(G_m)$.

The pointwise convergence of Fejér $(C, 1)$ means of functions on unbounded Vilenkin groups is one of the most celebrated problems in the field of dyadic harmonic analysis.

Fine [Fin] proved every Walsh–Fourier series (in the Walsh case $m_j = 2$ for all $j \in \mathbf{N}$) is a.e. (C, α) summable for $\alpha > 1$. His argument is an adaptation of the older trigonometric analogue due to Marcinkiewicz [Mar]. Schipp [Sch1] gave a simpler proof for the case $\alpha = 1$, i.e., $\sigma_n f \to f$ a.e. $(f \in L^1(G_m))$. He proved that σ^* is of weak type (L^1, L^1). That σ^* is of type (L^1, H^1) was discovered by Fujii [Fuj].

The theorem of Schipp and Fujii with respect to the character system of the group of 2-adic integers is proved by the author [Gát1].

The theorem of Schipp is generalized to the p-series fields ($m_j = p$ for all $j \in \mathbf{N}$) by Taibleson [Tai2] and later to bounded Vilenkin systems by Pál and Simon [PS]. The almost everywhere convergence $\sigma_n f \to f$ for integrable function f on noncommutative bounded Vilenkin groups and the (L^1, H^1) typeness of the maximal operator is proved by the author [Gát2]. We remark that the "noncommutative case" differs from the "commutative case" in many aspects. For instance there exist some bounded noncommutative Vilenkin groups in which the partial sums of the Fourier series do not converge to the function either in norm or a.e. for some $f \in L^p$, $p > 1$ [Gát2]. This is a sharp contrast.

With respect to unbounded Vilenkin groups nothing "positive" is yet known. The methods known in the trigonometric or in the Walsh, bounded Vilenkin case are not powerful enough. One of the main problems is that the proofs on the bounded Vilenkin groups (or in the trigonometric case) heavily use the fact that the L^1 norm of the Fejér kernels are uniformly bounded. This is not the case if the group G_m is an unbounded one [Pri]. From this it follows that the original theorem of Fejér does not hold on unbounded Vilenkin groups. Namely, Price proved [Pri] that for an arbitrary sequence m ($\sup_n m_n = \infty$) and $a \in G_m$ there exists a function f continuous on G_m and $\sigma_n f(a)$ does not converge to $f(a)$. Moreover, he proved [Pri] that if $((\log m_n)/M_n) \to \infty$, then there exist a function f continuous on G_m whose Fourier series are not $(C, 1)$ summable on a set $S \subset G_m$ which is non-denumerable. That is, only, a.e. convergence can be stated for unbounded Vilenkin groups.

THEOREM 1. *Let G_m be any Vilenkin group (bounded or not), $1 < p$ and $f \in L^p(G_m)$. Then, $\sigma_n f \to f$ $(n \to \infty)$ a.e. in G_m.*

Throughout this paper c denotes an absolute constant and c_p a constant which may depend only on p. Constants c, c_p may vary from line to line. The proof of Theorem 1 has several steps.

In Section 2 we discuss the boundedness of some operators on the discrete cyclic group Z_m $(2 \leqslant m \in \mathbf{N})$. This section is technical and will be used in the proof of Theorem 1 later, in Section 5. Section 2 is readable separately. In Section 3 we give an inequality $\mu(\{x \in G_m : \sigma^* f(x) > \lambda\}) \leqslant \mathbf{J}_1 + \mathbf{J}_{2,1} + \mathbf{J}_{2,2} + \mathbf{J}_{2,3}$. We prove that $\mathbf{J}_1 + \mathbf{J}_{2,1} \leqslant c \, \|f\|_1 / \lambda$ for all $\lambda > 0$, $f \in L^1(G_m)$. In the Section 4 (Lemma 4.4) the inequality

$$\mathbf{J}_{2,2} \leqslant c \frac{1}{\lambda} \|f\|_1 + c_p \frac{1}{\lambda^p} \|f\|_p^p$$

$(f \in L^p(G_m)$ $1 < p \leqslant 2)$ is verified. Section 5 is devoted to the confirmation of an upper bound like above for $\mathbf{J}_{2,3}$. This section concludes these investigations in order to prove Theorem 1.

2. SOME DISCRETE INEQUALITIES

Here the whole procedure is done on the discrete cyclic group Z_m $(2 \leqslant m \in \mathbf{P})$. Here we have $x, y \in Z_m$, $r(y) := \exp(2\pi \imath y/m)$, $2 \leqslant m \in \mathbf{P}$, $f : Z_m \to \mathbf{C}$. Note that (of course) $y - x(y + x)$ is the mod m subtraction (addition) of y and x. Also note that $\|f\|_p = (1/m)(\sum_{x=0}^{m-1} |f(x)|^p)^{1/p}$ $(1 \leqslant p < \infty)$ and $\mu_{Z_m}(A) = \text{card } A)/m$ for any $A \subset Z_m$.

The main aim of this section is to prove Theorem 2.8 (see the end of this section), that is, to prove that the maximal operator of the absolute values of the Fejér means of a functions is of type (L^p, L^p) $(1 < p < \infty)$ and of weak type (L^1, L^1) on the group Z_m (uniformly in m).

Set

$$T^{(1)} f(y) := \sup_{2 \leqslant k \in \mathbf{P}} \left| \frac{1}{m} \sum_{m/2 > x \geqslant m/k} f(y - x) \frac{r^k(x) - 1}{k(r(x) - 1)^2} \right|$$

and

$$T^{(2)}f(y) := \sup_{2 \leqslant k \in \mathbf{P}} \left| \frac{1}{m} \sum_{m - m/k \geqslant x \geqslant m/2} f(y-x) \frac{r^k(x)-1}{k(r(x)-1)^2} \right|.$$

Also set

$$R^- f(y) := \sup_{0 \neq k \in Z_m} \left| \frac{1}{k} \sum_{j=0}^{k-1} f(y-j) \right|,$$

$$R^+ f(y) := \sup_{0 \neq k \in Z_m} \left| \frac{1}{k} \sum_{j=0}^{k-1} f(y+j) \right|.$$

Let $0 < t \leqslant \pi/2$. Then use the inequality $ct \leqslant \sin t \leqslant t$ (for some absolute constant $c > 0$) throughout this section. We prove that operator R^-, R^+ is of type (L^p, L^p) on the group Z_m (uniformly in m) for all $1 < p \leqslant \infty$ and of weak type (L^1, L^1); that is,

LEMMA 2.1. *$\|Rf\|_p \leqslant c_p \|f\|_p$ $(1 < p \leqslant \infty)$ and $\mu_{Z_m}(\{y \in Z_m \colon Rf(y) > \lambda\}) \leqslant c \|f\|_1/\lambda$ for each $\lambda > 0$, where R is either R^- or R^+.*

We prove that operators $T^{(1)}$ and $T^{(2)}$ are (uniformly in m) of type (L^p, L^p) $(1 < p \leqslant \infty)$; that is,

LEMMA 2.2. *$\|T^{(j)}f\|_p \leqslant c_p \|f\|_p$ $(1 < p \leqslant \infty)$ and $\mu_{Z_m}(\{y \in Z_m \colon T^{(j)}f(y) > \lambda\}) \leqslant c \|f\|_1/\lambda$ for each $\lambda > 0$ $(j = 1, 2)$.*

In order to prove Lemma 2.1 we need the following decomposition lemma of type Calderon and Zygmund. $f \colon Z_m \to \mathbf{C}$, $\lambda > 0$.

LEMMA 2.3 [SWS, CZ]. *There exists a decomposition $f = f_0 + \sum_{i=1}^{\infty} f_i$, $|f_0| < c\lambda$, supp $f_i \subset [\alpha_i, \beta_i]$ disjoint intervals $(i \in \mathbf{P})$ of the interval $[0, m-1]$ $(= \{0, 1, ..., m-1\})$, where*

$$\frac{1}{\beta_i - \alpha_i + 1} \sum_{x=\alpha_i}^{\beta_i} |f_i(x)| < c\lambda, \quad \frac{1}{\beta_i - \alpha_i + 1} \sum_{x=\alpha_i}^{\beta_i} f_i(x) = 0 \qquad \text{for} \quad i \in \mathbf{P},$$

$$\mu_{Z_m}\left(\bigcup_{i \in \mathbf{P}} [\alpha_i, \beta_i] \right) = \sum_{i=1}^{\infty} \frac{\beta_i - \alpha_i + 1}{m} \leqslant c \|f\|_1/\lambda.$$

Proof of Lemma 2.1. We discuss the case $R = R^-$. The case $R = R^+$ can be treated in the same way and is left to the reader. The (L^∞, L^∞) typeness of R is trivial. Next, we prove that R is of weak type (L^1, L^1).

Let for $i \in \mathbf{P}$ $[\tilde{\alpha}_i, \tilde{\beta}_i] := [\alpha_i - (\beta_i - \alpha_i + 1), \beta_i + (\beta_i - \alpha_i + 1)]$ be the "tripled" interval, and the "tripled" of $F := \bigcup_{i \in \mathbf{P}} [\alpha_i, \beta_i]$ is $\tilde{F} := \bigcup_{i \in \mathbf{P}} [\tilde{\alpha}_i, \tilde{\beta}_i]$. Then

$$\mu_{Z_m}(Rf > c\lambda) \leqslant \mu_{Z_m}(\tilde{F}) + \mu_{Z_m}(\{ y : y \in Z_m \backslash \tilde{F}, Rf(y) > c\lambda\}) \\ =: J_1 + J_2.$$

By Lemma 2.3 we have $J_1 = \mu_{Z_m}(\tilde{F}) \leqslant c \, \|f\|_1 / \lambda$.

On the other hand, we prove for $y \in Z_m \backslash \tilde{F}$ that $Rf(y) \leqslant c\lambda$ (that is, $J_2 = 0$). $f = f_0 + \sum_{i=1}^{\infty} f_i =: f_0 + w$. $|f_0| < c\lambda$ implies that $Rf_0(y) \leqslant c\lambda$. Next, we prove $Rw(y) \leqslant c\lambda$. For a fixed $0 \neq k \in Z_m$,

$$\left| \frac{1}{k} \sum_{j=0}^{k-1} w(y-j) \right| \\ \leqslant \frac{1}{k} \sum_{\{i \in \mathbf{P} : [\alpha_i, \beta_i] \subset [y-k+1, y]\}} \sum_{\{j : y-j \in [\alpha_i, \beta_i]\}} |w(y-j)| \\ + \frac{1}{k} \sum_{\substack{\{i \in \mathbf{P} : [\alpha_i, \beta_i] \not\subset [y-k+1, y] \\ [\alpha_i, \beta_i] \cap [y-k+1, y] \neq \emptyset\},}} \sum_{\{j : y-j \in [\alpha_i, \beta_i] \cap [y-k+1, y]\}} |w(y-j)| \\ := J_{2,1} + J_{2,1}.$$

$$J_{2,1} \leqslant \frac{1}{k} \sum_{\{i \in \mathbf{P} : [\alpha_i, \beta_i] \subset [y-k+1, y]\}} \sum_{l \in [\alpha_i, \beta_i]} |f_i(l)| \\ \leqslant \frac{1}{k} \sum_{\{i \in \mathbf{P} : [\alpha_i, \beta_i] \subset [y-k+1, y]\}} c\lambda(\beta_i - \alpha_i + 1) \leqslant c\lambda,$$

because the intervals $[\alpha_i, \beta_i]$ $(i \in \mathbf{P})$ are disjoint and all of them are subsets of the interval $[y-k+1, y]$, the length of which is k.

Next, we give an upper bound for $J_{2,2}$. For a given $y \in Z_m \backslash \tilde{F}$ we have that only one of the disjoint intervals $[\alpha_i, \beta_i]$ can exist for which $[\alpha_i, \beta_i] \cap [y-k+1, y] \neq \emptyset$ and $[\alpha_i, \beta_i] \not\subset [y-k+1, y]$. Suppose that there is one, say, $[\alpha_1, \beta_1]$. Thus, function w can be different from 0 only on the interval $[\alpha_1, \beta_1]$. Since the distance of y from β_1 is not less than $\beta_1 - \alpha_1 + 1$, then $k \geqslant \beta_1 - \alpha_1 + 1$. Consequently,

$$J_{2,2} \leqslant \frac{1}{k} \sum_{\{j : y-j \in [\alpha_1, \beta_1]\}} |w(y-j)| = \frac{1}{k} \sum_{\{j : y-j \in [\alpha_1, \beta_1]\}} |f_1(y-j)| \\ \leqslant \frac{1}{k} (\beta_1 - \alpha_1 + 1) \, c\lambda \leqslant c\lambda.$$

That is, $Rf(y) \leqslant c\lambda$ on the set $y \in Z_m \backslash \tilde{F}$. This implies that R is of weak type (L^1, L^1). The interpolation theorem of Marcinkiewicz [SW, Zyg] gives that the operator R is of type (L^p, L^p) for all $1 < p \leqslant \infty$. Equality $R = R^-$

was supposed. The proof in the case of $R=R^+$ is the same; therefore it is left to the reader. ∎

Proof of Lemma 2.2. We deal with the case $j=1$. Since $m/2>x\geqslant m/k$,

$$\left|\frac{r^k(x)-1}{k(r(x)-1)^2}\right|\leqslant c\frac{1}{k\sin^2\pi(x/m)}\leqslant c\frac{m^2}{kx^2}.$$

Consequently,

$$\begin{aligned}T^{(1)}(y)&\leqslant c\frac{1}{m}\sup_{2\leqslant k\in Z_m}\sum_{m/2>x\geqslant m/k}|f(y-x)|\,c\frac{m^2}{kx^2}\\&\leqslant c\,\|f\|_\infty\sup_{2\leqslant k}\frac{m}{k}\sum_{m/2>x\geqslant m/k}\frac{1}{x^2}\leqslant c\,\|f\|_\infty.\end{aligned}$$

That is, operator $T^{(1)}$ is of type (L^∞, L^∞).

See the (L^p, L^p) typeness for $(1<p<\infty)$. First suppose that $f\geqslant 0$:

$$\begin{aligned}T^{(1)}(y)&\leqslant c\frac{1}{m}\sup_{2\leqslant k\in Z_m}\sum_{m/2>x\geqslant m/k}|f(y-x)|\,c\frac{m^2}{kx^2}\\&\leqslant c\sum_{n=0}^{\infty}\sup_{2\leqslant k\in Z_m}\sum_{2^{n+1(m/k)}>x\geqslant 2^{n(m/k)}}|f(y-x)|\,c\frac{m}{kx^2}\\&\leqslant c\sum_{n=0}^{\infty}\sup_{2\leqslant k\in Z_m}\frac{1}{2^n}\frac{k^2}{2^nm^2}\frac{m}{k}\sum_{2^{n+1(m/k)}>x\geqslant 2^{n(m/k)}}|f(y-x)|\\&\leqslant c\sum_{n=0}^{\infty}\frac{1}{2^n}\sup_{2\leqslant k\in Z_m}\frac{1}{2^{n(m/k)}}\sum_{2^{n+1(m/k)}\geqslant x>0}|f(y-x)|\\&\leqslant c\sum_{n=0}^{\infty}\frac{1}{2^n}\sup_{0\neq l\in Z_m}\frac{1}{l}\sum_{2l\geqslant x>0}|f(y-x)|\\&\leqslant cR^-f(y).\end{aligned}$$

This inequality and Lemma 2.1 give that operator $T^{(1)}$ is of weak type (L^1, L^1) for functions $f\geqslant 0$. Let f be a real function. Set

$$f^+(x):=\begin{cases}f(x) & \text{if } f(x)\geqslant 0\\ 0 & \text{if } f(x)<0\end{cases}\quad\text{and}\quad f^-(x):=\begin{cases}f(x) & \text{if } f(x)\leqslant 0\\ 0 & \text{if } f(x)>0\end{cases};$$

then $f=f^+-f^-$. The sublinearity of operator $T^{(1)}$ gives

$$\begin{aligned}\mu_{Z_m}(T^{(1)}f>2\lambda)&\leqslant\mu_{Z_m}(t^{(1)}f^+>\lambda)+\mu_{Z_m}(T^{(1)}f^->\lambda)\\&\leqslant c(\|f^+\|_1+\|f^-\|_1)/\lambda\leqslant c\,\|f\|_1/\lambda;\end{aligned}$$

that is, operator $T^{(1)}$ is of weak type (L^1, L^1) for real functions. If f: $Z_m \to \mathbb{C}$, then the $\Im f$ and $\Im f$ be the real and the imaginary part of f, respectively. Then,

$$\mu_{Z_m}(T^{(1)}f > 2\lambda) \leqslant \mu_{Z_m}(T^{(1)}\Re f > \lambda) + \mu_{Z_m}(T^{(1)}\Im f > \lambda)$$
$$\leqslant c(\|\Re f\|_1 + \|\Im f\|_1)/\lambda \leqslant c\,\|f\|_1/\lambda.$$

That is, operator $T^{(1)}$ is of weak type (L^1, L^1). After all, apply the interpolation theorem of Marcinkiewicz for the sublinear operator $T^{(1)}$. This gives that operator $T^{(1)}$ is of weak type (L^1, L^1) and of type (L^p, L^p) for all $1 < p \leqslant \infty$. In the case of the operator $T^{(2)}$ we do the same procedure with some minor modifications. Namely, set $x' := m - x$. Then,

$$T^{(2)}f(y) := \sup_{2 \leqslant k \in \mathbf{P}} \left| \frac{1}{m} \sum_{m/2 \geqslant x' \geqslant m/k} f(y + x') \frac{\bar{r}^k(x') - 1}{k(\bar{r}(x') - 1)^2} \right|.$$

Since

$$\left| \frac{\bar{r}^k(x') - 1}{k(\bar{r}(x') - 1)^2} \right| = \left| \frac{r^k(x') - 1}{k(r(x') - 1)^2} \right| \qquad (|\bar{z}| = |z| \text{ for all } z \in \mathbf{C}),$$

by the same method as in the case of $T^{(1)}$, but with R^+ instead of R^- the proof of Lemma 2.2 is complete. ∎

LEMMA 2.4.

$$\left| \frac{r^k(x) - 1}{k(r(x) - 1)^2} + \imath \frac{m}{2\pi x} \right| \leqslant ck,$$

where $0 < x < m/k$, $2 \leqslant k \in \mathbf{P}$, $x \in \mathbf{P}$.

Proof of Lemma 2.4. Let $0 < t \leqslant 2\pi$. Then,

$$|\cos t - 1| \leqslant \frac{t^2}{2!} + \frac{t^6}{6!} + \frac{t^{10}}{10!} + \cdots \leqslant ct^2 \tag{2.4.1}$$

and

$$|\sin t_t| \leqslant \frac{t^3}{3!} + \frac{t^7}{7!} + \frac{t^{11}}{11!} + \cdots \leqslant ct^3. \tag{2.4.2}$$

By (2.4.2) we have

$$\left|\frac{\imath\pi x/m}{2\sin^2(\pi x/m)} - \frac{\imath\pi x/m}{2(\pi x/m)^2}\right|$$

$$\leqslant c\,\frac{x}{m}\,\frac{|\sin(\pi x/m)+\pi x/m|\;|\sin(\pi x/m)-\pi x/m|}{\sin^2(\pi x/m)(\pi x/m)^2}$$

$$\leqslant c\,\frac{x}{m}\,\frac{(x/m)^3}{(x/m)^2\,(x/m)^2} \leqslant c. \tag{2.4.3}$$

Also by (2.4.2) we get

$$\left|\frac{\imath\sin(2\pi kx/m)}{4k\sin^2(\pi x/m)} - \frac{\imath\,2\pi kx/m}{4k\sin^2(\pi x/m)}\right| \leqslant c\,\frac{1}{k}\,(kx/m)^3\left(\frac{m}{x}\right)^2$$

$$\leqslant ck^2\,\frac{x}{m} \leqslant ck. \tag{2.4.4}$$

By (2.4.1) it follows

$$\left|\frac{\imath\sin(2\pi kx/m)}{4k}\left(\cot^2(\pi x/m) - \frac{1}{\sin^2(\pi x/m)}\right)\right|$$

$$\leqslant c\,\frac{1}{k}\,\frac{kx}{m}\,\frac{|\cos(\pi x/m)+1|\;|\cos(\pi x/m)-1|}{\sin^2(\pi x/m)}$$

$$\leqslant c\,\frac{1}{m}\,\frac{kx}{m}\,\frac{(x/m)^2}{(x/m)^2} \leqslant c\,\frac{x}{m} \leqslant c. \tag{2.4.5}$$

We also take into account

$$\left|\frac{\imath\sin(2\pi kx/m)}{k}\left(\frac{1}{2}+\frac{\imath}{2}\cot(\pi x/m)\right)^2 + \frac{\imath\sin(2\pi kx/m)}{4m}\cot^2(\pi x/m)\right|$$

$$\leqslant c\,\frac{kx/m}{k}\left(1+\frac{m}{x}\right) \leqslant c. \tag{2.4.6}$$

Note that

$$\frac{1}{1-r(x)} = \frac{1}{2} + \frac{\imath}{2}\cot \pi x/m.$$

Elementary calculations like above imply

$$\left|\frac{r^k(x)-1}{k(r(x)-1)^2}-\frac{\imath\sin(2\pi x/m)}{k(r(x)-1)^2}\right|\leqslant c\,\frac{|\cos(2\pi kx/m)-1|}{k(x/m)^2}$$
$$\leqslant c\,\frac{(kx/m)^2}{k(x/m)^2}\leqslant ck. \tag{2.4.7}$$

(2.4.3)–(2.4.7) imply

$$\begin{aligned}\left|\frac{r^k(x)-1}{k(r(x)-1)^2}+\imath\,\frac{m}{2\pi x}\right| &\leqslant \left|\frac{r^k(x)-1}{k(r(x)-1)^2}-\frac{\imath\sin(2\pi x/m)}{k(r(x)-1)^2}\right|\\ &\quad+\left|\frac{\imath\sin(2\pi kx/m)}{k}\left(\frac{1}{2}+\frac{\imath}{2}\cot(\pi x/m)\right)^2\right.\\ &\quad\left.+2\,\frac{\imath\sin(2\pi kx/m)}{4k}\cot^2(\pi x/m)\right|\\ &\quad+\left|-\frac{\imath\sin(2\pi kx/m)}{4k}\left(\cot^2(\pi x/m)-\frac{1}{\sin^2(\pi x/m)}\right)\right|\\ &\quad+\left|-\frac{\imath\sin(2\pi kx/m)}{4k\sin^2(\pi x/m)}+\frac{\imath 2\pi kx/m}{4k\sin^2(\pi x/m)}\right|\\ &\quad+\left|-\frac{\imath\pi x/m}{2\sin^2(\pi x/m)}+\frac{\imath\pi x/m}{2(\pi x/m)^2}\right|\\ &\quad+\left|-\frac{\imath\pi x/m}{2(\pi x/m)^2}+\imath\,\frac{m}{2\pi x}\right|\\ &\leqslant ck.\end{aligned}$$

This completes the proof of Lemma 2.4. ∎

Corollary 2.5.

$$\left|\frac{r^k(x)-1}{k(r(x)-1)^2}-\imath\,\frac{m}{2\pi(m-x)}\right|\leqslant ck,$$

where $m-(m/k)<x<m$, $2\leqslant k\in\mathbf{P}$, $x\in\mathbf{P}$.

Proof of Corollary 2.5. Let $t=m-x$. Then, $0<t<m/k$, $1/(1-r(x))=1/2+(\imath/2)\cot(\pi x/m)=(1/2)-(\imath/2)\cot(\pi t/m)$,

$$\left|\frac{r^k(x)-1}{k(r(x)-1)^2}+\frac{r^k(t)-1}{k(r(t)-1)^2}\right|$$
$$\leqslant \frac{1}{k}\left(|r^k(x)-1|\left|\frac{1}{4}+\frac{\imath}{2}\cot(\pi x/m)\right|\right.$$
$$\left.+|r^k(t)-1|\left|\frac{1}{4}+\frac{\imath}{2}\cot(\pi t/m)\right|\right)$$
$$+\frac{1}{k}\left|\frac{-1}{4}\cot^2(\pi x/m)(r^k(x)+r^k(t)-1)\right|$$
$$\leqslant c\,\frac{1}{k}\left|\sin(\pi kt/m)\right|(1+m/t)+c\,\frac{1}{k}\left(\frac{m}{t}\right)^2 2\,|\cos(2\pi kt/m)-1|$$
$$\leqslant c\,\frac{1}{k}\frac{kt}{m}\left(1+\frac{m}{t}\right)+c\,\frac{1}{k}\left(\frac{m}{t}\right)^2\left(\frac{kt}{m}\right)^2$$
$$\leqslant ck.$$

This and Lemma 2.4 give that the proof of Corollary 2.5 is complete. ∎

Set

$$T_k^{(3)}f(y):=\frac{1}{m}\sum_{\substack{0<x<m/k\\ m-m/k<x<m}} f(y-x)\,\frac{r^k(x)-1}{k(r(x)-1)^2},$$

$$T_k^{(4)}f(y):=\sum_{\{x\,:\,m/k>|y-x|>0\}} f(x)\,\frac{1}{y-x}$$

for $1<k\in Z_m$. We prove

LEMMA 2.6.

$$\left|T_k^{(3)}f(y)+\frac{\imath}{2\pi}\,T_k^{(4)}f(y)\right|\leqslant c(R^+\,|f|\,(y)+R^-\,|f|\,(y)).$$

Proof. By Lemma 2.4 we have

$$\frac{1}{m}\left|\sum_{0<x<m/k} f(y-x)\left(\frac{r^k(x)-1}{k(r(x)-1)^2}+\frac{\imath m}{2\pi x}\right)\right|$$
$$\leqslant c\,\frac{k}{m}\sum_{0<x<m/k}|f(y-x)|\leqslant cR^-\,|f|\,(y).$$

Meanwhile, by Corollary 2.5 we have

$$\frac{1}{m}\left|\sum_{m-m/k<x<m} f(y-x)\left(\frac{r^k(x)-1}{k(r(x)-1)^2}-\frac{\imath m}{2\pi(m-x)}\right)\right|$$
$$\leqslant c\,\frac{k}{m}\sum_{m-m/k<x<m} |f(y-x)|$$
$$= c\,\frac{k}{m}\sum_{0<t<m/k} |f(y+t)| \leqslant R^+\,|f|\,(y)$$

$(t := m-x,\ f(y-x) = f(y+t-m = f(y+t))$. This and

$$\sum_{0<x<m/k} f(y-x)\,\frac{1}{x} = \sum_{y-m/k<t<y} f(t)\,\frac{1}{y-},$$

$$\sum_{m-m/k<x<m} f(y-x)\,\frac{-1}{m-x} = \sum_{0<s<m/k} f(y+s-m)\,\frac{-1}{s}$$
$$= \sum_{0<s<m/k} f(y+s)\,\frac{-1}{s}$$
$$= \sum_{y<t<y+m/k} f(t)\,\frac{-1}{t-y}$$
$$= \sum_{y<t<y+m/k} f(t)\,\frac{1}{y-t},$$

that is,

$$T_k^{(4)} f(y) = \sum_{y-m/k<x<y} f(x)\,\frac{1}{y-x} + \sum_{y<x<y+m/k} f(x)\,\frac{1}{y-x}$$
$$= \sum_{0<x<m/k} f(y-x)\,\frac{1}{x} + \sum_{m-m/k<x<m} f(y-x)\,\frac{-1}{m-x}$$

$(1<k\in Z_m)$, gives that the proof of Lemma 2.6 is complete. ▮

Set $T^{(j)}f := \sup_{1<k\in Z_m} |T_k^{(j)} f|$ for $j=3, 4$. By Lemmas 2.1, 2.6 and by the so-called maximal Hilbert transform (on the unit interval) (see, e.g., [BS]) we prove

LEMMA 2.7. *The operator* $T^{(3)}$ *is of weak type* (L^1, L^1) *and of type* (L^p, L^p) (*for all* $1<p<\infty$) *on the discrete group* Z_m.

Proof. Denote by ν the Lebesgue measure on the unit interval $[0, 1)$. Let $g: [0, 1) \to \mathbf{C}$ be integrable with respect to ν. Define the maximal Hilbert transform of g as

$$\mathscr{H} g(s) := \sup_{\varepsilon > 0} \left| \int_{\{t \in [0,1) : |s-t| > \varepsilon\}} g(t) \frac{1}{s-t} \, d\nu(t) \right|.$$

It is known (see, e.g., [BS]) that $\mathscr{H}$ is of type (L^p, L^p) (for all $1 < p < \infty$) and of weak type (L^1, L^1) on $[0, 1)$ with respect to ν. We apply this for the function

$$g(t) := f(j) \qquad \text{for} \quad \frac{j}{m} \leqslant t < \frac{j+1}{m} \quad (j \in Z_m).$$

Let $s, t \in [0, 1)$, $|s - t| \geqslant 1/m$:

$$\left| \frac{1}{s-t} - \frac{1}{\frac{[ms]}{m} - \frac{[mt]}{m}} \right| \leqslant \frac{2/m}{|s-t| \left| \frac{[ms]}{m} - \frac{[mt]}{m} \right|} \leqslant c \frac{m}{([ms] - [mt])^2}.$$

($[t]$ is the integer part of real number t.) Thus,

$$\begin{aligned} &\int_{\{t \in [0,t) : |s-t| \geqslant 1/m\}} |g(t)| \left| \frac{1}{s-t} - \frac{1}{\frac{[ms]}{m} - \frac{[mt]}{m}} \right| d\nu(t) \\ &\leqslant c \sum_{\substack{j=0 \\ j \neq [ms]}}^{m-1} \int_{j/m}^{(j+1)/m} |f(j)| \frac{m}{([ms] - j)^2} \, d\nu(t) \\ &\leqslant c \sum_{\substack{j=0 \\ j \neq [ms]}}^{m-1} |f(j)| \frac{1}{([ms] - j)^2}. \end{aligned} \tag{2.7.1}$$

(2.7.1) gives

$$\begin{aligned} &\int_0^1 \int_{\{t \in [0,1) : |s-t| \geqslant 1/m\}} |g(t)| \left| \frac{1}{s-t} - \frac{1}{\frac{[ms]}{m} - \frac{[mt]}{m}} \right| d\nu(t) \, d\nu(s) \\ &\leqslant c \frac{1}{m} \sum_{k=0}^{m-1} \sum_{j=0,\, j \neq k}^{m-1} |f(j)| \frac{1}{(k-j)^2} \\ &\leqslant c \frac{1}{m} \sum_{j=0}^{m-1} |f(j)| = c \, \|f\|_1. \end{aligned}$$

Since

$$\int_{\{t \in [0,1) : |s-t| \geqslant 1/m\}} g(t) \frac{1}{\frac{[ms]}{m} - \frac{[mt]}{m}} \, d\nu(t) \geqslant T_k^{(4)} f([ms]),$$

thus

$$\begin{aligned} \mu_{Z_m}(y : \sup_{0<k \in Z_m} |T_k^{(4)} f(y)| > \lambda) &= \nu(s : \sup_{0<k\in Z_m} |T_k^{(4)} f([ms])| > \lambda) \\ &\leqslant \nu(s : \mathscr{H} g(s) > \lambda/2) + \frac{c}{\lambda} \|f\|_1 \\ &\leqslant c \frac{1}{\lambda} \int_0^1 |g(s)| \, d\nu(s) + \frac{c}{\lambda} \|f\|_1 \\ &\leqslant \frac{c}{\lambda} \|f\|_1. \end{aligned}$$

Similarly, for $1 < p < \infty$,

$$\begin{aligned} \|\sup_k |T_k^{(4)} f| \|_p &\leqslant \left(\int_0^1 |\mathscr{H} g(s)|^p \, d\nu(s) \right)^{1/p} + c \, \|f\|_p \\ &\leqslant c_p \|f\|_p. \end{aligned}$$

By Lemma 2.6 and Lemma 2.1 (R^+, R^- is of type (L^p, L^p) $(1 < p \leqslant \infty)$ and of weak type (L^1, L^1)) we complete the proof of Lemma 2.7. ∎

Since the kth Fejér kernel on the discrete group Z_m $(1 < k \in Z_m)$ is

$$\begin{aligned} K_k^{(2_m)}(x) &:= \frac{1}{k} \sum_{j=0}^{k-1} \sum_{l=0}^{j-1} r^l(x) \\ &= \begin{cases} \dfrac{r^k(x) - 1}{k(r(x)-1)^2} - \dfrac{1}{r(x)-1} & \text{if} \quad 2 \leqslant k \leqslant m, \\ 0 & \text{if} \quad k = 0, 1, \end{cases} \end{aligned}$$

and since the discrete Hilbert transform

$$Hf(y) := \frac{1}{m} \left| \sum_{x=0, x \neq y}^{m-1} f(x) \frac{1}{r(y-x)-1} \right|$$

is of type (L^p, L^p) $(1 < p < \infty)$ and of weak type (L^1, L^1) (on the group Z_m) (see, e.g., [BS; Zyg; EG, p. 120; Sub]), by Lemmas 2.2, 2.7 we proved

THEOREM 2.8. *The maximal operators*

$$\sigma^{*(Z_m)}f(y) := \frac{1}{m} \sup_{0<k\in Z_m} \left| \sum_{x=0,\, x\neq y}^{m-1} f(x)\, K_k^{(Z_m)}(y-x) \right|,$$

$$T^{(Z_m)}f(y) := \frac{1}{m} \sup_{0<k\in Z_m} \left| \sum_{x=0,\, x\neq y}^{m-1} f(x) \frac{r^k(y-x)-1}{k(r(y-x)-1)^2} \right|$$

are of type (L^p, L^p) $(1<p<\infty)$ *and of weak type* (L^1, L^1) *on the group* Z_m.

More accurately, we discussed only the case $2\leqslant k$, but the case $k=1$ is trivial. (If $k=1$, then

$$K_k^{(Z_m)} = 0, \qquad \frac{r^k - 1}{k(r-1)^2} = \frac{1}{r-1};$$

the kernel of the discrete Hilbert transform.)

3. SOME INITIAL INVESTIGATION OF THE PROOF OF THEOREM 1

For $l, L \in \mathbf{N}$ define the sum of Dirichlet kernel functions

$$K_{l,L} := \sum_{i=l}^{l+L-1} D_i.$$

Let $n^{(j)} := \sum_{i=j}^{\infty} n_i M_i$ $(n, j\in\mathbf{N})$ and for $n\in\mathbf{N}$ set $|n| := \max(i\in\mathbf{N}: n_i\neq 0)$. That is, $|n| = A$ if and only if $M_A \leqslant n < M_{A+1}$. Then by elementary calculations we have

$$nK_n = \sum_{s=0}^{|n|} \sum_{j=0}^{n_s-1} K_{n^{(s+1)}+jM_s,\, M_s}. \tag{3.1}$$

By (3.1) we have

$$\mu(\sigma^* f > \lambda) = \mu\left(y : \sup_{n\in\mathbf{P}} \left| \int_{G_m} f(x)\, K_n(y-x)\, d\mu(x) \right| > \lambda \right)$$

$$\leqslant \mu\left(y : \sup_{n\in\mathbf{P}} \left| \int_{I_{|n|+1}(y)} f(x)\, K_n(y-x)\, d\mu(x) \right| > \lambda/2 \right)$$

$$+\mu\left(y : \sup_{n\in\mathbf{P}} \frac{1}{n}\left|\sum_{t=0}^{|n|}\int_{I_t(y)\backslash I_{t+1}(y)} f(x)\right.\right.$$

$$\left.\left.\times\sum_{s=0}^{|n|}\sum_{j=0}^{n_s-1} K_{n^{s+1}+jM_s, M_s}(y-x)\,d\mu(x)\right| > \lambda/2\right)$$

$$=: \mathbf{J}_1 + \mathbf{J}_2.$$

Discuss $\mathbf{J}_2$:

$$\mathbf{J}_2 \leqslant \mu\left(y\colon \sup_{n\in\mathbf{P}} \frac{1}{n}\left|\sum_{t=1}^{|n|}\sum_{s=0}^{t-1}\int_{I_t(y)\backslash I_{t+1}(y)} f(x)\right.\right.$$

$$\left.\left.\times\sum_{j=0}^{n_s-1} K_{n^{s+1}+jM_s, M_s}(y-x)\,d\mu(x)\right| > \lambda/6\right)$$

$$+\mu\left(y : \sup_{n\in\mathbf{P}} \frac{1}{n}\left|\sum_{s=1}^{|n|}\sum_{t=0}^{s-1}\int_{I_t(y)\backslash I_{t+1}(y)} f(x)\right.\right.$$

$$\left.\left.\times\sum_{j=0}^{n_s-1} K_{n^{s+1}+jM_s, M_s}(y-x)\,\mu(x)\right| > \lambda/6\right)$$

$$+\mu\left(y : \sup_{n\in\mathbf{P}} \frac{1}{n}\left|\sum_{t=0}^{|n|}\int_{I_t(y)\backslash I_{t+1}(y)} f(x)\right.\right.$$

$$\left.\left.\times\sum_{j=0}^{n_t-1} K_{n^{(t+1)}+jM_t, M_t}(y-x)\,d\mu(x)\right| > \lambda/6\right)$$

$$=: \mathbf{J}_{2,1} + \mathbf{J}_{2,2} + \mathbf{J}_{2,3}.$$

In this section we give an upper bound for $\mathbf{J}_{2,1}$ and for $\mathbf{J}_1$. Let $t\in\mathbf{N}$, $z\in I_t\backslash I_{t+1}$. Then for $s<t$ by (1.2) we have

$$|K_{n^{(s+1)}+jM_s, M_s}(z)| \leqslant c n_t M_t M_s.$$

Thus,

$$\mu\left(y\in G_m : \sup_{n\in\mathbf{P}} \frac{1}{n}\sum_{t=1}^{|n|}\sum_{s=0}^{t-1}\sum_{s=0}^{t-1}\left|\int_{I_t(y)\backslash I_{t+1}(y)} f(x)\right.\right.$$

$$\left.\left.\times\sum_{j=0}^{n_s-1} K_{n^{(s+1)}+jM_s, M_s}(y-x)\,d\mu(s)\right| > \lambda\right)$$

$$\leqslant \mu\left(y\in G_m : c\sup_{n\in\mathbf{P}}\sum_{t=1}^{|n|}\sum_{s=0}^{t-1}\int_{I_t(y)\backslash I_{t+1}(y)} |f(x)|\, n_s M_s n_t M_t\,d\mu(x) > \lambda\right)$$

$$\leqslant \mu\left(y\in G_m : c\int_{G_m}\sup_{n\in\mathbf{P}}\frac{1}{n}\sum_{t=1}^{|n|} n_t M_t M_t \int_{I_t(y)} |f(x)|\, d\mu(x) > \lambda\right)$$

$$\leqslant \mu\left(y\in G_m : c\sup_{n\in\mathbf{P}}\frac{1}{n}\sum_{t=1}^{|n|} n_t M_t\, |f|^*(y) > \lambda\right)$$

$$\leqslant \mu(y\in G_m : c\,|f|^*(y) > \lambda)$$

$$\leqslant c\, \||f|\|_1/\lambda = c\,\|f\|_1/\lambda.$$

For the inequality $\mu(y\in G_m: f^*(y)>\lambda)\leqslant c\,\|f\|_1/\lambda$ see the paper of Burkholder [Bur]. This is, $\mathbf{J}_{2,1}\leqslant c\,\|f\|_1/\lambda$. Moreover, by $|K_n|\leqslant (n+1)/2$ we have

$$\mu\left(y\in G_m : \sup_{n\in\mathbf{P}}\left|\int_{I_{|n|+1}(y)} f(x)\, K_n(y-x)\, d\mu(x)\right| > \lambda\right)$$

$$\leqslant \mu\left(y\in G_m : \sup_{n\in\mathbf{P}}\int_{I_{|n|+1}(y)} |f(x)|\,\frac{n+1}{2}\, d\mu(x) > \lambda\right)$$

$$\leqslant \mu\left(y\in G_m : c\sup_{n\in\mathbf{P}}\frac{n+1}{2M_{|n|+1}}\,|f|^*(y) > \lambda\right)$$

$$\leqslant c\,\|f\|_1/\lambda.$$

That is, $\mathbf{J}_1\leqslant c\,\|f\|_1/\lambda$.

4. THE CASE $S>T$

For $n, s, t\in\mathbf{N}$ and $|n|\geqslant s>t$, $y\in G_m$ set $e_t := (0, 0, ..., 0, 1, 0, ...)$, where the tth coordinate of e_t is 1 and the rest are zeros, and set

$$T_n^{s,t} f(y) := \frac{1}{m_t}\left|\sum_{x_t=0,\, x_t\neq y_t}^{m_t-1} E_s(f\bar{\psi}_{n^{(s)}})(y+e_t(x_t-y_t))\,\frac{1}{1-r_t(y-x)}\right|.$$

(This means that $E_s(f\bar{\psi}_{n^{(s)}})(y+e_t(x_t-y_t))$ depends on $y_0, y_1, ..., y_{t-1}, x_t, y_{t+1}, ..., y_{s-1}$ (and on n, s, f of course) and we sum with respect to x_t.) Since $n^{(s)}$ does not depend on $n_0, ..., n_{s-1}$, we can introduce the notation

$$\sum_{n^{(s)}} = \sum_{\{n\in\mathbf{N}:\, n_0=n_1=\cdots=n_{s-1}=0\}} = \sum_{n_s=0}^{m_s-1}\sum_{n_{s+1}=0}^{m_{s+1}-1}\cdots,$$

and

$$\sup_{n^{(s)}} = \sup_{\{n \in \mathbf{N} : n_0 = n_1 = \cdots = n_{s-1} = 0\}}.$$

Then we prove

Lemma 4.1. *$\|\sup_{n^{(s)}} T_n^{s,t} f\|_p \leqslant c_p \|f\|_p$, where $f \in L^p(G_m)$, $1 < p \leqslant 2$.*

Proof of Lemma 4.1. Set $y' = y + e_t(x_t - y_t)$ $(x_t = 0, ..., m_t - 1)$. Apply the Hausdorff–Young inequality (see, e.g., [Zyg, BS]) for the Vilenkin group $\times_{k=s}^{\infty} Z_{m_k}$ and for the function

$$\frac{1}{m_t} \sum_{x_t = 0,\, x_t \neq y_t}^{m_t - 1} f(y') \frac{1}{1 - r_t(y - x)}.$$

Let $1/p + 1/q = 1$:

$$\left(\sum_{n^{(s)}} \left| E_s \left(\left(\frac{1}{m_t} \sum_{x_t = 0,\, x_t \neq y_t}^{m_t - 1} f(y') \frac{1}{1 - r_t(y - x)} \right) \bar{\psi}_{n^{(s)}}(y) \right)(y) \right|^q \right)^{1/q}$$
$$\leqslant \left(E_s \left(\left| \frac{1}{m_t} \sum_{x_t = 0,\, x_t \neq y_t}^{m_t - 1} f(y') \frac{1}{1 - r_t(y - x)} \right|^p \right)(y) \right)^{1/p}.$$

The discrete Hilbert transform is of type $(L^p(Z_{m_t}), L^p(Z_{m_t}))$ (see, e.g., [Sub; EG, p. 120]) for all $1 < p < \infty$; consequently,

$$\|\sup_{n^{(s)}} T_n^{s,t} f\|_p \leqslant \left\| \left(\sum_{n^{(s)}} |T_n^{s,t} f|^q \right)^{1/q} \right\|_p$$
$$\leqslant \left(E_0 \left(E_s \left(\left| \frac{1}{m_t} \sum_{x_t = 0,\, x_t \neq y_t}^{m_t - 1} f(y') \frac{1}{1 - r_t(y - x)} \right|^p \right) \right) \right)^{1/p}$$
$$= \left(E_0 \left(\left| \frac{1}{m_t} \sum_{x_t = 0,\, x_t \neq y_t}^{m_t - 1} f(y') \frac{1}{1 - r_t(y - x)} \right|^p \right) \right)^{1/p}$$
$$= \left(E_0 \left(\frac{1}{m_t} \sum_{y_t = 0}^{m_t - 1} \left| \frac{1}{m_t} \sum_{x_t = 0,\, x_t \neq y_t}^{m_t - 1} f(y') \frac{1}{1 - r_t(y - x)} \right|^p \right) \right)^{1/p}$$
$$\leqslant c_p \left(E_0 \left(\frac{1}{m_t} \sum_{x_t = 0}^{m_t - 1} |f(y')|^p \right) \right)^{1/p}$$
$$= c_p \|f\|_p. \quad \blacksquare$$

Let $j \in \mathbf{N}$, $A, k \in \mathbf{P}$ be fixed. Let $n \in \mathbf{P}$ and $|n| = A \geqslant j + k$. Set

$$T_A f(y) := \sup_{n^{(A-j)},\ |n|=1} |T_n^{A-j,\, A-j-k} f(y)|$$

$$= \sup_{\{n \in \mathbf{P} : n_0 = 0, \ldots, n_{A-j-1} = 0,\ |n| = A\}} |T_n^{A-j,\, A-j-k} f(y)|$$

$$= \sup_{\{n \in \mathbf{P} : |n| = A\}} |T_n^{A-j,\, A-j-k} f(y)|,$$

$$Tf(y) := \sup_{\{A \in \mathbf{P} : A \geqslant j+k\}} T_A f(y)$$

for $y \in G_m$. Then we prove

Lemma 4.2.

$$\mu(Tf > \lambda) \leqslant \left(c \frac{1}{\lambda} \|f\|_1 + c_p \left(\frac{1}{\lambda} \right)^p \|f\|_p^p \right) (j+k)$$

for all $\lambda > 0$, $f \in L^p(G_m)$, $1 < p \leqslant 2$.

Proof of Lemma 4.2. Define the stopping time ν (see, e.g., [Sch2]) as

$$\nu(x) := \inf(k \in \mathbf{N} : E_k(|f|)(x) > \lambda) \qquad (\inf \emptyset = \infty).$$

It is known [Sch2] that $\mu(\nu < \infty) \leqslant \|f\|_1 / \lambda$. Denote the characteristic function of the set $B \subset G_m$ by 1_B, i.e.,

$$1_B(x) := \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \notin B \end{cases}.$$

Since $T_n^{s,t} f = T_n^{s,t}(E_{A+1} f) = T_n^{s,t}(E_{A+1} f - E_A f)$, then $T_A f = T_A(E_{A+1} f) = T_A(E_{A+1} f - E_A f)$ ($|n| = A$). This and

$$1 = 1_{G_m}$$

$$= 1_{\{\nu < \infty\}} + 1_{\{\nu = \infty\}}$$

$$= 1_{\{\nu > A+1\}} + 1_{\{\nu < A-j-k\}} + 1_{\{A-j-k \leqslant \nu \leqslant A+1\}}$$

give

$$\begin{aligned}
\mu(Tf>\lambda) & \\
&\leqslant \mu(1_{\{v<\infty\}} Tf<\lambda/2) \\
&\quad + \mu(1_{\{v=\infty\}} \sup_{\{A\in\mathbf{P}:\, A\geqslant j+k\}} A_A(A_{\{v>A+1\}}(E_{A+1}f-E_Af))>\lambda/6) \\
&\quad + \mu(1_{\{v=\infty\}} \sup_{\{A\in\mathbf{P}:\, A\geqslant j+k\}} T_A(1_{\{v<A-j-k\}}(E_{A+1}f-E_Af))>\lambda/6) \\
&\quad + \mu(1_{\{v=\infty\}} \sup_{\{A\in\mathbf{P}:\, A\geqslant j+k\}} T_A(1_{\{A-j-k\leqslant v\leqslant A+1\}}(E_{A+1}f-E_Af)) \\
&\quad > \lambda/6) \\
&=: J_1+J_2+J_3+J_4.
\end{aligned}$$

We already have $J_1\leqslant c\,\|f\|_1/\lambda$.

$J_3=0$ is given by

$$\begin{aligned}
0 &\leqslant T_A(1_{\{v<A-j-k\}}f) \\
&= \sup_{n^{(A-j)},\, |n|=A} 1_{\{v<A-j-k\}}\, |T_n^{A-j,\,A-j-k} f| \\
&\leqslant \sup_{n^{(A-j)},\, |n|=A} 1_{\{v<A-j-k\}} \sup_{n^{(A-j)},\, |n|=A} |T_n^{A-j,\,A-j-k} f| = 0
\end{aligned}$$

on the set $\{v=\infty\}$ and so does $\sup_{\{A\in\mathbf{P}:\, A\geqslant j+k\}} T_A(1_{\{v<A-j-k\}}f)$.

Apply Lemma 4.1 with $p=2$ in order to have an upper bound for J_2:

$$\begin{aligned}
J_2 &\leqslant c\,\frac{1}{\lambda^2}\sum_{A\in\mathbf{N}} \|T_A(1_{\{v>A+1\}}(E_{A+1}f-E_Af))\|_2^2 \\
&\leqslant c\,\frac{1}{\lambda^2}\sum_{A\in\mathbf{N}} \|1_{\{v>A+1\}}(E_{A+1f-E_Af})\|_2^2.
\end{aligned}$$

The lemma of Burkholder [Bur, Sch2] gives that

$$\sum_{A\in\mathbf{N}} \|1_{\{v>A+1\}}(E_{A+1}f-E_Af)\|_2^2 \leqslant c\,\|f\|_1\,\lambda.$$

That is, $J_2\leqslant c\,\|f\|_1/\lambda$.

Give an upper bound for J_4:

$$J_4 \leqslant \frac{1}{\lambda^p} \left\| \left(\sum_{A=j+k}^{\infty} \sup_{n^{(A-j)},\ |n|=1} |T_n^{A-j,\, A-j-k}(1_{\{A-j-k \leqslant v \leqslant A+1\}} \times (E_{A+1}f - E_A f))|^p \right)^{1/p} \right\|_p^p$$

$$= \frac{1}{\lambda^p} \sum_{A=j+k}^{\infty} \left\| \sup_{n^{(A-j)},\ |n|=1} T_n^{A-j,\, A-j-k}(1_{\{A-j-k \leqslant v \leqslant A+1\}} \times (E_{A+1}f - E_A f)) \right\|_p^p$$

$$\leqslant \frac{c}{\lambda^p} \sum_{A=j+k}^{\infty} \left\| \sup_{n^{(A-j)},\ |n|=A} T_n^{A-j,\, A-j-k}(1_{\{A-j-k \leqslant v \leqslant A+1\}} E_{A+1}f) \right\|_p^p + \frac{c}{\lambda^p} \sum_{A=j+k}^{\infty} \left\| \sup_{n^{(A-j)},\ |n|=A} T_n^{A-j,\, A-j-k}(1_{\{A-j-k \leqslant v \leqslant A+1\}} E_A f) \right\|_p^p$$

$$=: J_{4,1} + J_{4,2}.$$

Apply again Lemma 4.1:

$$J_{4,1} \leqslant c_p \frac{1}{\lambda^p} \sum_{A \geqslant j+k} \int_{G_m} 1_{\{A-j-k \leqslant v \leqslant A+1\}} |E_{A+1}f|^p$$

$$= c_p \frac{1}{\lambda^p} \sum_{A \geqslant j+k} \sum_{l=A-j-k}^{A+1} \int_{\{v=l\}} |E_{A+1}f|^p$$

$$= c_p \frac{1}{\lambda^p} \sum_{l=0}^{\infty} \sum_{A=l-1}^{l+j+k} \int_{\{v=l\}} |E_{A+1}f|^p$$

$$\leqslant c_p \frac{1}{\lambda^p} \sum_{l=0}^{\infty} (j+k+2) \int_{\{v=l\}} (f^*)^p$$

$$\leqslant c_p \frac{1}{\lambda^p} (j+k+2) \|f^*\|_p^p$$

$$\leqslant c_p \frac{1}{\lambda^p} (j+k+2) \|f\|_p^p$$

(the last inequality is due to Burkholder [Bur]). In the same way we have

$$J_{4,2} \leqslant c_p \frac{1}{\lambda p} (j+k+2) \|f\|_p^p.$$

That is, the proof of Lemma = 4.2 is complete. ▮

LEMMA 4.3. *Let* $|n| \geqslant s > t$, $n, s, t \in \mathbf{N}$, $z \in I_t \backslash I_{t+1}$. *Then,*

$$K_{n^{(s)} M_s}(z) = \begin{cases} 0, & \text{if } \; z - e_t z_t \notin I_s \\ M_t M_s \psi_{n^{(s)}}(z) \dfrac{1}{1 - r_t(z)}, & \text{if } \; z - e_t z_t \in I_s. \end{cases}$$

Proof. By (1.2) we have

$$\begin{aligned} K_{n^{(s)}, M_s}(z) &= \sum_{k=n^{(s)}}^{n^{(s)}+M_s-1} D_k(z) \\ &= \sum_{k=n^{(s)}}^{n^{(s)}+M_s-1} \psi_k(z) \left(\sum_{j=0}^{t-1} k_j M_j + M_t \sum_{i=m_t-k_t}^{m_t-1} r_t^i(z) \right) \\ &=: J_1 + J_2. \end{aligned}$$

First we prove that $J_1 = 0$. Since $z \in I_t$, then $\psi_k(z) = \prod_{l=0}^{\infty} r_l^{k_l}(z) = \prod_{l=t}^{\infty} r_l^{k_l}(z) = \psi_{k^{(t)}}(z)$. Next,

$$J_1 = \sum_{l=0}^{M_s-1} \psi_{n^{(s)}+l}(z) \sum_{j=0}^{t-1} l_j M_j = \sum_{l=0}^{M_s-1} \psi_{n^{(s)}+l^{(t)}}(z) \sum_{j=0}^{t-1} l_j M_j$$

because $(n^{(s)}+l)^{(t)} = n^{(s)} + l^{(t)} (l < M_s, s > t)$. Thus,

$$J_1 = \sum_{l_o=0}^{m_o-1} \cdots \sum_{l_{t-1}=0}^{m_{t-1}-1} \sum_{l_{t+1}=0}^{m_{t+1}-1} \cdots \sum_{l_{s-1}=0}^{m_{s-1}-1} \psi_{n^{(s)}}(z) \sum_{j=0}^{t-1} l_j M_j \sum_{l_t=0}^{m_t-1} \psi_{l^{(t)}}(z) = 0,$$

because $\sum_{l_t=0}^{m_t-1} \psi_{l^{(t)}}(z) = \psi_{l^{(t+1)}}(z) \sum_{l_t=0}^{m_t-1} r_t^{l_t}(z) = 0$ ($z_t \neq 0$ since $z \in I_t \backslash I_{t+1}$). That is, $J_1 = 0$. Next discuss J_2:

$$\begin{aligned} J_2 &= M_t \sum_{k=n^{(s)}}^{n^{(s)}+M_s-1} \psi_{k^{(t+1)}}(z)\, r_t^{k_t}(z) \sum_{i=m_t-k_t}^{m_t-1} r_t^i(z) \\ &= M_t \sum_{k=n^{(s)}}^{n^{(s)}+M_s-1} \psi_{k^{(t+1)}}(z) \frac{r_t^{k_t}(z) - 1}{r_t(z) - 1} \\ &= M_t \sum_{l=0}^{M_s-1} \psi_{n^{(s)}+l^{(t+1)}}(z) \frac{r_t^{l_t}(z) - 1}{r_t(z) - 1} \end{aligned}$$

$$= M_t \psi_{n^{(s)}}(z) \sum_{l=0}^{M_s-1} \psi_{l^{(t)}}(z) \frac{1}{r_t(z)-1}$$

$$+ M_t \psi_{n^{(s)}}(z) \sum_{l=0}^{M_s-1} \psi_{l^{(t+1)}}(z) \frac{1}{1-r_t(z)}$$

$$=: J_{2,1} + J_{2,2}.$$

Since $\psi_l(z = \psi_{l^{(t)}}(z)$ for $z \in I_t$,

$$J_{2,1} = M_t \psi_{n^{(s)}}(z) \frac{1}{r_t(z)-1} \sum_{l=0}^{M_s-1} \psi_l(z) = 0.$$

On the other hand, if $z - e_t z_t \notin I_s$, then there exists an $a \in \{t+1, t+2, ..., s-1\}$ for which $z - e_t z_t \in I_a \backslash I_{a+1}$. Then, since $\psi_{l^{(t+1)}}(z) = \psi_{l^{(t+1)}}(z - z_t e_t) = \psi_l(z - z_t e_t)$ $(z - z_t e_t \in I_{t+1})$ we have

$$\sum_{l=0}^{M_s-1} \psi_{l^{(t+1)}}(z) = \sum_{l=0}^{M_s-1} \psi_{l^{(t+1)}}(z - z_t e_t) = 0.$$

That is, in this case $J_{2,2} = 0$ also.

If $z - e_t z_t \in I_s$, then

$$J_{2,2} = M_t M_s \psi_{n^{(s)}}(z) \frac{1}{1-r_t(z)}.$$

This completes the proof of Lemma 4.3. ∎

LEMMA 4.4.

$$J := \mu\left(y : \left| \sup_{n \in \mathbf{P}} \frac{1}{n} \sum_{s=1}^{|n|} \sum_{t=0}^{s-1} \int_{I_t(y) \backslash I_{t+1}(y)} f(x) \right.\right.$$

$$\left.\left. \times \sum_{i=0}^{n_s-1} K_{n^{(s+1)}+iM_s, M_s}(y-x)\, d\mu(x) \right| > \lambda \right)$$

$$\leqslant c \frac{1}{\lambda} \|f\|_1 + c_p \frac{1}{\lambda^p} \|f\|_p^p$$

for all $f \in L^p(G_m) 1 < p \leqslant 2.$

Proof. Let $n, j \in \mathbf{N}$, $k \in \mathbf{P}$ be fixed and denote by $s := |n| - j$, $t := |n| - j - k$. Then by Lemma 4.3 we have

$$\begin{aligned}
&\frac{1}{n}\left|\int_{I_t(y)\setminus I_{t+1}(y)} f(x)\sum_{i=0}^{n_s-1} K_{n^{(s+1)}+iM_s,\,M_s}(y-x)\,d\mu(x)\right|\\
&\quad=\frac{M_{t+1}M_s}{n}\left|\frac{1}{m_t}\sum_{b=1}^{m_t-1}\int_{I_s(e_t b)} f(x)\,\psi_{n^{(s+1)}}(y-x)\,\frac{1}{1-r_t(y-x)}\right.\\
&\qquad\left.\times\sum_{i=0}^{n_s-1} r_s^i(y-x)\,d\mu(x)\right|\\
&\quad\leqslant \sup_{n_s\in Z_{m_s}} \frac{M_{t+1}n_s}{n}\,|T_n^{s,t}f(y)|\\
&\quad=: J_1.
\end{aligned}$$

If $j\geqslant 1$, then

$$\frac{M_{t+1}n_s}{n}\leqslant\frac{M_{t+1}m_s}{M_{|n|}}\leqslant c2^{-j-k}.$$

If $j=0$, then

$$\frac{M_{t+1}n_s}{n}=\frac{M_{t+1}n_{|n|}}{n}\leqslant\frac{M_{t+1}n_{|n|}}{n_{|n|}M_{|n|}}\leqslant c2^{-k}=c2^{-j-k}$$

also in this case. Thus,

$$J_1\leqslant c2^{-j-k}\sup_{\substack{n\in\mathbf{P}\\|n|\geqslant j+k}}|T_n^{|n|-j,\,|n|-j-k}f(y)|.$$

Then, by Lemma 4.2 it follows

$$\begin{aligned}
J&\leqslant\mu\left(y: c\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}2^{-j-k}\sup_{\substack{n\in\mathbf{P}\\|n|\geqslant j+k}}|T_n^{|n|-j,\,|n|-j-k}f(y)|>\lambda\right)\\
&\leqslant\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\mu\left(y=c2^{(-j-k)/2}\sup_{\substack{n\in\mathbf{P}\\|n|\geqslant j+k}}|T_n^{|n|-j,\,|n|-j-k}f(y)|>\lambda\right)\\
&\leqslant\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\left(c2^{(-j-k)/2}\frac{1}{\lambda}\|f\|_1+c_p2^{((-j-k)/2)\,p}\frac{1}{\lambda^p}\|f\|_p^p\right)(j+k)\\
&\leqslant c\frac{1}{\lambda}\|f\|_1+c_p\frac{1}{\lambda^p}\|f\|_p^p.
\end{aligned}$$

That is, the proof of Lemma 4.4 is complete. ∎

5. THE CASE $s=t$ (THE SUBCASE $|n|>t$)

Let $z \in I_t \backslash I_{t+1}$. Then, by (1.2) we have

$$\begin{aligned} K_{n^{(t+1)}+jM_t,\, M_t} &= \sum_{k=n^{(t+1)}+jM_t}^{n^{(t+1)}+(j+1)M_t-1} \psi_{n^{(t+1)}+jM_t}(z) \left(\sum_{l=0}^{t-1} k_l M_l \right) \\ &\quad + \sum_{k=n^{(t+1)}+jM_t}^{n^{(t+1)}+(j+1)M_t-1} \psi_{n^{(t+1)}+jM_t}(z)\, M_t \left(\sum_{l=m_t-j}^{m_t-1} r_t^l(z) \right) \\ &:= A_1 + A_2. \end{aligned}$$

Since

$$A_1(z) = \psi_{n^{(t+1)}+jM_t}(z) \frac{M_t(M_t-1)}{2},$$

we have $|A_1| \leqslant c M_t^2$ and

$$\begin{aligned} &\sup_{n\in\mathbf{P}} \left| \frac{1}{n} \sum_{t=0}^{|n|-1} \sum_{j=0}^{n_t-1} \int_{I_t(y)\backslash I_{t+1}(y)} f(x)\, A_1(y-x)\, d\mu(x) \right| \\ &\qquad \leqslant c \sup_{n\in\mathbf{P}} \frac{1}{n} \sum_{t=0}^{|n|-1} m_t M_t E_t(|f|)(y) \leqslant c\, |f|^*\,(y). \qquad (5.1) \end{aligned}$$

On the other hand,

$$\begin{aligned} A_2(z) &= \psi_{n^{(t+1)}}(z)\, M_t^2 \frac{r_t^j(z)-1}{r_t(z)-1}, \\ \sum_{j=0}^{n_t-1} A_2(z) &= \psi_{n^{(t+1)}}(z)\, M_t^2 \frac{r_t^{n_t}(z)-1}{(r_t(z)-1)^2} - \psi_{n^{(t+1)}}(z)\, M_t M_t n_t \frac{1}{r_t(z)-1} \\ &=: A_{2,1} - A_{2,2}. \end{aligned}$$

By Section 4, more precisely by Lemma 4.2, we have

$$\begin{aligned} &\mu\left(y : \sup_{n\in\mathbf{P}} \frac{1}{n} \sum_{t=0}^{|n|-1} \left| \int_{I_t(y)\backslash I_{t+1}(y)} f(x)\, A_{2,2}(y-x)\, d\mu(x) \right| > \lambda \right) \\ &\quad = \mu\left(y : \sup_{n\in\mathbf{P}} \frac{1}{n} \sum_{t=0}^{|n|-1} M_t M_t n_t \left| \int_{I_t(y)\backslash I_{t+1}(y)} f(x)\, \psi_{n^{(t+1)}}(y-x) \right.\right. \\ &\qquad \left.\left. \times \frac{1}{r_t(y-x)-1}\, d\mu(x) \right| > \lambda \right) \end{aligned}$$

$$\leqslant \mu\left(y : \sup_{n\in\mathbf{P}} \frac{1}{n} \sum_{t=0}^{|n|-1} M_t M_{t+1} \left| \int_{I_t(y)\setminus I_{t+1}(y)} f(x)\, \psi_{n^{(t+1)}}(y-x)\right.\right.$$

$$\left.\left. \times \frac{1}{r_t(y-x)-1}\, d\mu(x)\right| > \lambda\right)$$

$$\leqslant \mu\left(y : c \sup_{n\in\mathbf{P}} \sum_{t=0}^{|n|-1} 2^{t-|n|}\, |T_n^{t+1,\,t} f(y)| > \lambda\right)$$

$$= \mu\left(y : c \sup_{n\in\mathbf{P}} \sum_{j=0}^{|n|-1} 2^{-j-1}\, |T_n^{|n|-j,\,|n|-j-1} f(y)| > \lambda\right)$$

$$\leqslant c \sum_{j=0}^{\infty} \mu\left(y : \sup_{n\in\mathbf{P},\,|n|\geqslant j+1} 2^{-j/2}\, |T_n^{|n|-j,\,|n|-j-1} f(y)| > c\lambda\right)$$

$$\leqslant \sum_{j=0}^{\infty} c\left(\frac{2^{-j/2}}{\lambda}\, \|f\|_1 + c_p \left(\frac{2^{-j/2}}{\lambda}\right)^p \|f\|_p^p\right) j$$

$$\leqslant c\, \frac{1}{\lambda}\, \|f\|_1 + c_p\, \frac{1}{\lambda^p}\, \|f\|_p^p \tag{5.2}$$

for $1 < p \leqslant 2$. Let $1 < p \leqslant 2$, $1/p + 1/q = 1$. We give an upper bound for

$$\mu\left(y : \sup_{n\in\mathbf{P}} \frac{1}{n} \sum_{t=0}^{|n|-1} \left| \int_{I_t(y)\setminus I_{t+1}(y)} f(x)\, A_{2,1}(y-x)\, d\mu(x)\right| > \lambda\right)$$

$$\leqslant \mu\left(y : \sup_{n\in\mathbf{P}} \frac{M_{t+1}}{M_{|n|}} \sup_{n^{(t)}} \sum_{t=0}^{|n|-1} \left| M_t \int_{I_t(y)\setminus I_{t+1}(y)} f(x)\, \bar{\psi}_{n^{(t+1)}}(x)\right.\right.$$

$$\left.\left. \times \frac{r_t^{n_t}(y-x)-1}{n_t(r_t(y-x)-1)^2}\, d\mu(x)\right|\right)$$

$(M_t \leqslant M_{t+1}/n_t)$. Set, like in Lemma 4.1,

$$T_n^t f(y) := \frac{1}{m_t} \left| \sum_{x_t=0,\, x_t\neq y_t}^{m_t-1} E_{t+1}(f\bar{\psi}_{n^{(t+1)}})(y + e_t(x_t - y_t))\right.$$

$$\left. \times \frac{r_t^{n_t}(y-x)-1}{n_t(r_t(y-x)-1)^2}\right|.$$

Apply the Hausdorff–Young inequality (see, e.g., [SWS, Zyg]) for the function

$$g(x + e_t(y_t - x_t)) := \frac{1}{m_t} \sum_{x_t=0,\, x_t\neq y_t}^{m_t-1} f(x)\, \frac{r_t^{n_t}(y-x)-1}{n_t(r_t(y-x)-1)^2},$$

and for the Vilenkin group $\times_{k=t}^{\infty} Z_{m_k}$; that is, we have

$$\left(\sum_{n^{(t+1)}} |E_{t+1}(g(x+e_t(y_t-x_t))\,\bar{\psi}_{n^{(t+1)}}(x))|^q\right)^{1/q} \leqslant (|E_{t+1}(g(x+e_t(y_t-x_t)))|^p)^{1/p}.$$

Consequently, by Theorem 2.8 we get

$$\begin{aligned}\|\sup_{n^{(t)}} T_n^t\|_p &\leqslant \left\|\sup_{n_t}\left(\sum_{n^{(t+1)}} |T_n^t f|^q\right)^{1/q}\right\|_p \\ &\leqslant \|\sup_{n_t} (|E_{t+1}(g(x+e_t(y_t-x_t)))|^p)^{1/p}\|_p \\ &\leqslant c_p\,\|f\|_p.\end{aligned}$$

Let $j, A \in \mathbf{P}$ be fixed. Let $n \in \mathbf{P}$ and $|n| = A > j$. Set (like in Lemma 4.2)

$$T_A f(y) := \sup_{n^{(A-j)},\ |n|=A} |T_n^{A-j} f(y)|, \qquad Tf(y) := \sup_{A\in\mathbf{P},\ A>j} T_A f(y)$$

for $y \in G_m$. Then we prove

$$\mu(Tf > \lambda) \leqslant \left(c\,\frac{1}{\lambda}\,\|f\|_1 + c_p\left(\frac{1}{\lambda}\right)^p \|f\|_p^p\right) j.$$

The proof of this inequality is the same as the proof of Lemma 4.2 (exactly, step by step) and is left to the reader. From this it follows

$$\begin{aligned}&\mu\left(y : \sup_{n\in\mathbf{P}} \frac{1}{n}\sum_{t=0}^{|n|-1}\left|\int_{I_t(y)\setminus I_{t+1}(y)} f(x)\,A_{2,1}(y-x)\,d\mu(x)\right| > \lambda\right) \\ &= \mu\left(y : \sup_{n\in\mathbf{P}} \frac{1}{n}\sum_{t=0}^{|n|-1} M_t M_t n_t \left|\int_{I_t(y)\setminus I_{t+1}(y)} f(x)\,\psi_{n^{(t+1)}}(y-x)\right.\right. \\ &\qquad \left.\left. \times \frac{r_t^{n_t}(y-x)-1}{n_t(r_t(y-x)-1)^2}\,d\mu(x)\right| > \lambda\right) \\ &\leqslant \mu\left(y : \sup_{n\in\mathbf{P}} \frac{1}{n}\sum_{t=0}^{|n|-1} M_{t+1}\,|T_n^t f(y)| > \lambda\right) \\ &\leqslant \mu\left(y : c\sum_{j=1}^{\infty} 2^{-j} \sup_{n\in\mathbf{P}\ |n|>j} |T_n^{|n|-j} f(y)| > \lambda\right)\end{aligned}$$

$$\leqslant \sum_{j=1}^{\infty} \mu\Big(y : c2^{-j/2} \sup_{n \in \mathbf{P}\ |n| > j} |T_n^{|n|-j} f(y)| > \lambda\Big)$$

$$\leqslant \sum_{j=1}^{\infty} j\left(c\,\frac{1}{2^{j/2}\lambda}\,\|f\|_1 + c_p\,\frac{1}{(2^{j/2}\lambda)^p}\,\|f\|_p^p\right)$$

$$\leqslant c\,\frac{1}{\lambda}\,\|f\|_1 + c_p\,\frac{1}{\lambda^p}\,\|f\|_p^p \tag{5.3}$$

(For $1 < p \leqslant 2$).

Lemma 5.1. *Let operator T_n° be linear ($n \in \mathbf{N}$) with property*

$$T_n^\circ f = T_n^\circ(E_{|n|+1} f), \qquad T_n^\circ(fg) = g T_n^\circ f$$

for g is $\mathscr{A}_{|n|}$ measurable and $|T_n^\circ(1)| \leqslant c$ ($1 : G_m \to \mathbf{C}$ is given by $1(x) := 1$ ($x \in G_m$)). Suppose that operator $T_A := \sup_{n \in \mathbf{N},\ |n| = A} |T_n^\circ|$ is of type $(L^2(G_m),\ L^2(G_m))$ and $(L^p(G_m), L^p(G_m))$ for some $1 < p < \infty$ $A \in \mathbf{N}$ (uniformly in A). Then, for $T := \sup_{A \in \mathbf{N}} T_A$ we have

$$\mu(Tf > \lambda) \leqslant c\,\frac{\|f\|_1}{\lambda} + cp\,\frac{\|f\|_p^p}{\lambda^p}$$

for each $f \in L^p(G_m)$.

Proof. The properties of operator T_A ($A \in \mathbf{N}$) give

$$T_A f = \sup_{n \in \mathbf{N},\ |n| = A} |T_n^\circ f|$$

$$\leqslant \sup_{n,\ \in \mathbf{N},\ |n| = A} |T_n^\circ(E_{|n|+1} f - E_{|n|} f)| + \sup_{n \in \mathbf{N},\ |n| = A} |T_n^\circ(E_{|n|} f)|$$

$$= \sup_{n \in \mathbf{N},\ |n| = A} |T_n^\circ(E_{A+1} f - E_A f)| + \sup_{n \in \mathbf{N},\ |n| = A} |T_n^\circ(E_{|n|} f)|$$

$$\leqslant T_A(E_{A+1} f - E_A f) + |E_A f| \sup_{n \in \mathbf{N},\ |n| = A} |T_n^\circ(1)|$$

$$\leqslant T_A(E_{A+1} f - E_A f) + c\,|E_A f|.$$

Let ν be the stopping time, as in the proof of Lemma 4.2. We follow the proof of Lemma 4.2:

$$1 = 1_{G_m} = 1_{\{\nu < \infty\}} + 1_{\{\nu = \infty\}} = 1_{\{\nu > A+1\}} + 1\{\nu \leqslant A\} + 1_{\{\nu = A+1\}}.$$

This and the sublinearity of T_A, T give

$$\begin{aligned}\mu(Tf>\lambda) &\leqslant \mu(1_{\{\nu<\infty\}}\,Tf>\lambda/2)\\ &\quad+\mu(1_{\{\nu=\infty\}}\sup_{A\in\mathbf{N}}\,(T_A(E_{A+1}f-E_Af)+c\,|E_Af|)>\lambda/2)\\ &\leqslant \mu(1_{\{\nu<\infty\}}\,Tf>\lambda/2)\\ &\quad+\mu(1_{\{\nu=\infty\}}\sup_{A\in\mathbf{N}}\,T_A(1_{\{\nu>A+1\}}(E_{A+1}f-E_Af))>\lambda/12)\\ &\quad+\mu(1_{\{\nu=\infty\}}\sup_{A\in\mathbf{N}}\,T_A(1_{\{\nu\leqslant A\}}(E_{A+1}f-E_Af))>\lambda/12)\\ &\quad+\mu(1_{\{\nu=\infty\}}\sup_{A\in\mathbf{N}}\,T_A(1_{\{\nu=A+1\}}(E_{A+1}f-E_Af))>\lambda/12)\\ &\quad+\mu(1_{\{\nu=\infty\}}\,c\,|E_Af|>\lambda/4)\\ &=: J_1+J_2+J_3+J_4+J_5.\end{aligned}$$

The definition of ν given $J_1\leqslant c\,\|f\|_1/\lambda$. We have also $J_5\leqslant c\,\|f\|_1/\lambda$. $J_3=0$ is given by

$$\begin{aligned}0&\leqslant 1_{\{\nu=\infty\}}\sup_{A\in\mathbf{N}}\,T_A(1_{\{\nu\leqslant A\}}(E_{A+1}f-E_Af))\\ &=1_{\{\nu=\infty\}}\sup_{A\in\mathbf{N}}\ \sup_{n\in\mathbf{N},\,|n|=A}\ T_n^{\circ}(1_{\{\nu\leqslant A\}}(E_{A+1}f-E_Af))\\ &=1_{\{\nu=\infty\}}\sup_{A\in\mathbf{N}}\ \sup_{n\in\mathbf{N},\,|n|=A}\ (1_{\{\nu\leqslant A\}}\,T_n^{\circ}(E_{A+1}f-E_Af))\\ &=1_{\{\nu=\infty\}}\sup_{A\in\mathbf{N}}(1_{\{\nu\leqslant A\}}\,T_A(E_{A+1}f-E_Af))\\ &\leqslant 1_{\{\nu=\infty\}}\sup_{A\in\mathbf{N}}1_{\{\nu\leqslant A\}}\ \sup_{A\in\mathbf{N}}\,T_A(E_{A+1}f-E_Af)=0.\end{aligned}$$

Since T_A is of type (L^2, L^2), in the very same way as in the proof of Lemma 4.2 we get

$$J_2\leqslant c\,\frac{1}{\lambda^2}\sum_{A\in\mathbf{N}}\|1_{\{\nu>A+1\}}(E_{A+1}f-E_Af)\|_2^2\leqslant c\,\|f\|_1/\lambda.$$

All that remains is to discuss J_4. Since $1<p<\infty$, we have

$$\begin{aligned} J_4 &\leqslant c_p \frac{1}{\lambda^p} \sum_{A\in\mathbf{N}} \|T_A(1_{\{v=A+1\}}(E_{A+1}f-E_Af))\|_p^p \\ &\leqslant c_p \frac{1}{\lambda^p} \sum_{A\in\mathbf{N}} \left(\int_{G_m} 1_{\{v=A+1\}}\,|E_{A+1}f|^p+\int_{G_m} 1_{\{v=A+1\}}\,|E_Af|^p\right) \\ &\leqslant c_p \frac{1}{\lambda^p} \sum_{A\in\mathbf{N}} \int_{\{v=A+1\}} (f^*)^p \\ &\leqslant c_p \frac{1}{\lambda^p}\,\|f^*\|_p^p \\ &\leqslant c_p \frac{1}{\lambda^p}\,\|f\|_p^p. \end{aligned}$$

This completes the proof of Lemma 5.1. ∎

The Subcase $|n|=t$

Let $z\in I_t\backslash I_{t+1}$. Then, by (1.2) we have

$$\begin{aligned} &K_{n^{(t+1)}+jM_t,\,M_t} \\ &\quad= K_{jM_{|n|},\,M_{|n|}}(z) \\ &\quad= \sum_{k=jM_{|n|}}^{(j+1)\,M_{|n|}-1} r_{|n|}^j(z)\left(\sum_{l=0}^{|n|-1} k_lM_l+M_{|n|}\sum_{l=m_{|n|}-j}^{m_{|n|}-1} r_{|n|}^l(z)\right) \\ &\quad=: A_3(z)+A_4(z). \end{aligned}$$

Then,

$$A_3(z)=\frac{M_{|n|}(M_{|n|}-1)}{2}\,r_{|n|}^j(z)$$

and

$$\begin{aligned} &\sup_{n\in\mathbf{P}}\left|\frac{1}{n}\sum_{j=0}^{n_{|n|}-1}\int_{I_{|n|}(y)\backslash I_{|n|+1}(y)} f(x)\,A_3(y-x)\,d\mu(x)\right| \\ &\qquad\leqslant c\sup_{n\in\mathbf{P}}\frac{n_{|n|}M_{|n|}^2}{n_{|n|}\,M_{|n|}}\int_{I_{|n|}(y)\backslash I_{|n|+1}(y)} |f(x)|\,d\mu(x)\leqslant c\,|f|^*(y). \qquad (5.4) \end{aligned}$$

On the other hand,

$$A_4(z) = M_{|n|}^2 \frac{r_{|n|}^j(z)-1}{r_{|n|}(z)-1},$$

$$\sum_{j=0}^{n_{|n|}-1} A_4(z) = M_{|n|}^2 \frac{r_{|n|}^{n_{|n|}}(z)-1}{(r_{|n|}(z)-1)^2} + M_{|n|} M_{|n|} n_{|n|} \frac{1}{1-r_{|n|}(z)}$$

$$=: A_{4,1} + A_{4,2}.$$

Apply Theorem 2.8 and Lemma 5.1 for the operator

$$T_n^\circ f(y) := \frac{1}{m_{|n|}} \sum_{x_{|n|}=0,\, x_{|n|}\neq y_{|n|}}^{m_{|n|}-1} E_{|n|+1} f(x) \frac{r_{|n|}^{n_{|n|}}(y-x)-1}{n_{|n|}(r_{|n|}(y-x)-1)^2}$$

$$(n \in \mathbf{N}).$$

Note that the condition $|T_n^\circ(1)| \leqslant c$ holds, because

$$\left| \frac{1}{m_{|n|}} \sum_{x_{|n|}=1}^{M_{|n|}-1} \frac{r_{|n|}^{n_{|n|}}(x)-1}{n_{|n|}(r_{|n|}(x)-1)^2} \right| \leqslant c.$$

(This holds because

$$\frac{1}{m_{|n|}} \left| \sum_{x_{|n|}=1}^{m_{|n|}-1} K^{Z_{m_n}}(x) \right| \leqslant c, \qquad \frac{1}{m_{|n|}} \left| \sum_{x_{|n|}=1}^{m_{|n|}-1} \frac{1}{1-r_{|n|}(x)} \right| \leqslant c.)$$

It follows $(n \geqslant n_{|n|} M_{|n|})$

$$\mu\left(y : \sup_{n\in\mathbf{P}} \frac{1}{n} \left| \int_{I_{|n|}(y)\setminus I_{|n|+1}(y)} f(x)\, A_{4,1}(y-x)\, d\mu(x) \right| > \lambda \right)$$

$$\leqslant \mu\left(y : \sup_{n\in\mathbf{P}} M_{|n|} \left| \int_{I_{|n|}(y)\setminus I_{|n|+1}(y)} f(x) \right.\right.$$

$$\left.\left. \times \frac{r_{|n|}^{n_{|n|}}(y-x)-1}{r_{|n|}(r_{|n|}(y-x)-1)^2}\, d\mu(x) \right| > \lambda \right)$$

$$\leqslant c \frac{1}{\lambda} \|f\|_1 + c_p \frac{1}{\lambda^p} \|f\|_p^p \tag{5.5}$$

for each $f \in L^p(G_m)$ $1<p<\infty$. The (L^p, L^p) typeness (for all $1<p<\infty$) of the discrete Hilbert transform and Lemma 5.1 give

$$\mu\left(y : \sup_{n \in \mathbf{P}} \frac{1}{n} \left| \int_{I_{|n|}(y) \setminus I_{|n|+1}(y)} f(x)\, A_{4,2}(y-x)\, d\mu(x) \right| > \lambda \right)$$

$$\leqslant \mu\left(y : \sup_{n \in \mathbf{P}} M_{|n|} \left| \int_{I_{|n|}(y) \setminus I_{|n|+1}(y)} f(x) \frac{1}{1 - r_{|n|}(y-x)}\, d\mu(x) \right| > \lambda \right)$$

$$\leqslant c \frac{1}{\lambda} \|f\|_1 + c_p \frac{1}{\lambda^p} \|f\|_p^p. \tag{5.6}$$

Summing up our achievements in this section by (5.1), (5.2), (5.3), (5.4), (5.5), and (5.6) we get

$$\mu\left(y : \sup_{n \in \mathbf{P}} \frac{1}{n} \left| \sum_{t=0}^{|n|} \int_{I_t(y) \setminus I_{t+1}(y)} f(x) \right.\right.$$

$$\left.\left. \times \sum_{j=0}^{n_t - 1} K_{n^{(t+1)} + jM_t, M_t}(y-x)\, d\mu(x) \right| > \lambda \right)$$

$$\leqslant \mu\left(y : \sup_{n \in \mathbf{P}} \frac{1}{n} \left| \sum_{t=0}^{|n|-1} \int_{I_t(y) \setminus I_{t+1}(y)} f(x) \sum_{j=0}^{n_t - 1} A_1(y-x)\, \mu(x) \right| > \lambda/6 \right)$$

$$+ \mu\left(y : \sup_{n \in \mathbf{P}} \frac{1}{n} \left| \sum_{t=0}^{|n|-1} \int_{I_t(y) \setminus I_{t+1}(y)} f(x)\, A_{2,2}(y-x)\, \mu(x) \right| > \lambda/6 \right)$$

$$+ \mu\left(y : \sup_{n \in \mathbf{P}} \frac{1}{n} \left| \sum_{t=0}^{|n|-1} \int_{I_t(y) \setminus I_{t+1}(y)} f(x)\, A_{2,1}(y-x)\, \mu(x) \right| > \lambda/6 \right)$$

$$+ \mu\left(y : \sup_{n \in \mathbf{P}} \frac{1}{n} \left| \int_{I_{|n|}(y) \setminus I_{|n|+1}(y)} f(x) \right.\right.$$

$$\left.\left. \times \sum_{j=0}^{n_{|n|} - 1} A_3(y-x)\, d\mu(x) \right| > \lambda/6 \right)$$

$$+ \mu\left(y : \sup_{n \in \mathbf{P}} \frac{1}{n} \left| \int_{I_{|n|}(y) \setminus I_{|n|+1}(y)} f(x)\, A_{4,1}(y-x)\, d\mu(x) \right| > \lambda/6 \right)$$

$$+ \mu\left(y : \sup_{n \in \mathbf{P}} \frac{1}{n} \left| \int_{I_{|n|}(y) \setminus I_{|n|+1}(y)} f(x)\, A_{4,2}(y-x)\, d\mu(x) \right| > \lambda/6 \right)$$

$$\leqslant \mu(|f|^* > c\lambda) + \left(c\frac{1}{\lambda}\|f\|_1 + \frac{1}{\lambda^p}\|f\|_p^p\right) + \left(c\frac{1}{\lambda}\|f\|_1 + c_p\frac{1}{\lambda^p}\|f\|_p^p\right)$$

$$+ \mu(|f|^* > c\lambda) + \left(c\frac{1}{\lambda}\|f\|_1 + c_p\frac{1}{\lambda^p}\|f\|_p^p\right)$$

$$+ \left(c\frac{1}{\lambda}\|f\|_1 + c_p\frac{1}{\lambda^p}\|f\|_p^p\right)$$

$$\leqslant c\frac{1}{\lambda}\|f\|_1 + c_p\frac{1}{\lambda^p}\|f\|_p^p. \tag{5.7}$$

The Final Step of the Proof of Theorem 1

Recall the notation of Section 3 ($\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_{2,1}, \mathbf{J}_{2,2}, \mathbf{J}_{2,3}$). In the Section 3 we proved that

$$\mathbf{J}_1 + \mathbf{J}_{2,1} \leqslant \frac{1}{\lambda} c\|f\|_1$$

for all $f \in L^1(G_m)$. In the Section 4 (Lemma 4.4) the inequality

$$\mathbf{J}_{2,2} \leqslant c\frac{1}{\lambda}\|f\|_1 + c_p\frac{1}{\lambda^p}\|f\|_p^p$$

for every $f \in L^p(G_m)$ $1 < p \leqslant 2$ is verified. This by (5.7) (here one can find an upper bound for $\mathbf{J}_{2,3}$) means that

$$\mu(\sigma^* f > \lambda) \leqslant \mathbf{J}_1 + \mathbf{J}_{2,1} + \mathbf{J}_{2,2} + \mathbf{J}_{2,3}$$
$$\leqslant c\frac{1}{\lambda}\|f\|_1 + c_p\frac{1}{\lambda^p}\|f\|_p^p$$

for each $f \in L^p(G_m)$ $1 < p \leqslant 2$.

After all let $v > 1$ and $f \in L^v(G_m)$. Then there exists $1 < p \leqslant 2$ for which $f \in L^p(G_m)$ (e.g., $\min(2, v)$). Let $\varepsilon > 0$ be arbitrary. Then there exists a Vilenkin polynomial P ($= \sum_{j=0}^k a_j \psi_j$ for some $a_0, ..., a_k \in \mathbf{C}$ $(k \in \mathbf{N})$) for which $\|f - P\|_p < \varepsilon$. Consequently, since $\lim_n \sigma_n P = P$ everywhere, by the given bound for $\mu(\sigma^* f > \lambda)$ it follows for $\delta > 0$ that

$$\mu(\overline{\lim_{n\in\mathbf{P}}}\,|\sigma_n f-f|>\delta)\leqslant\mu(\overline{\lim_{n\in\mathbf{P}}}\,|\sigma_n f-\sigma_n P|>\delta/3)$$

$$+\mu(\overline{\lim_{n\in\mathbf{P}}}\,|\sigma_n P-P|>\delta/3)+\mu(\overline{\lim_{n\in\mathbf{P}}}\,|P-f|>\delta/3)$$

$$\leqslant c\frac{1}{\delta}\,\|f-P\|_1+c_p\frac{1}{\delta^p}\,\|f-P\|_p^p$$

$$\leqslant c\frac{1}{\delta}\,\varepsilon+c_p\frac{1}{\delta^p}\,\varepsilon^p.$$

Letting $\varepsilon\to 0$ we have

$$\mu(\overline{\lim_{n\in\mathbf{P}}}\,|\sigma_n f-f|>\delta)=0$$

for any $\delta>0$. This means that $\sigma_n f\to f$ a.e. $(n\to\infty)$. The proof of Theorem 1 is complete. ▮

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